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TOTALLY REAL SUBMANIFOLDS IN $S^6(1)$ SATISFYING CHEN'S EQUALITY

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Abstract. In this paper, we study 3-dimensional totally real submanifolds of $S^6(1)$. If this submanifold is contained in some 5-dimensional totally geodesic $S^5(1)$, then we classify such submanifolds in terms of complex curves in $\mathbb{C}P^2(4)$ lifted via the Hopf fibration $S^5(1) \to \mathbb{C}P^2(4)$. We also show that such submanifolds always satisfy Chen's equality, i.e. $\delta_M = 2$, where $\delta_M(p) = \tau(p) - \inf K(p)$ for every $p \in M$. Then we consider 3-dimensional totally real submanifolds which are linearly full in $S^6(1)$ and which satisfy Chen's equality. We classify such submanifolds as tubes of radius $\pi/2$ in the direction of the second normal space over an almost complex curve in $S^6(1)$.

1. Introduction

In the last 15 years, many authors have studied submanifolds of the nearly Kaehler 6-sphere $S^6(1)$. Almost complex curves in $S^6(1)$ have been studied amongst others by Bryant [Br] and Sekigawa [S]. In a recent paper [BVW1], Bolton, Vrancken and Woodward have divided the almost complex curves in 4 classes. One of the classes is characterized by the fact that they lie linearly full in some 5-dimensional totally geodesic $S^5(1)$. This class can be described using totally real surfaces in the complex projective space $\mathbb{C}P^2(4)$ via the Hopf fibration $S^5(1) \to \mathbb{C}P^2(4)$. Totally real minimal surfaces in $S^6(1)$ are now completely classified by the same authors in [BVW2]. None of them lie linearly full in $S^6(1)$.

In this paper, we study 3-dimensional totally real submanifolds of $S^6(1)$. Ejiri proved in [E1] that such submanifolds are automatically minimal. Other results about such submanifolds are obtained in [E2], [M], [DVV1] and [DVV2]. Here we first assume that the submanifold is contained in some 5-dimensional totally geodesic $S^5(1)$. We classify such submanifolds in terms of complex curves in $\mathbb{C}P^2(4)$ lifted via the Hopf fibration $S^5(1) \to \mathbb{C}P^2(4)$, see Theorem 1 and Theorem 4 below.

On the other hand, B.Y. Chen has given in [C] a best possible inequality between the sectional curvature K, the scalar curvature τ, which we somewhat nonstandardly define by $\tau = \sum_{i<j} K(e_i \wedge e_j)$, and the mean curvature function of a submanifold M^n of a real space of constant curvature c. For that purpose, a new Riemannian invariant δ_M was introduced by

$$\delta_M(p) = \tau(p) - \inf K(p),$$

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where $\inf K$ is the function assigning to each $p \in M$ the infimum of $K(\pi)$, where π runs over all planes in T_pM. Then Chen's inequality is given by

$$\delta_M \leq \tfrac{n^2(n-2)}{2(n-1)}\|H\|^2 + \tfrac{1}{2}(n+1)(n-2)c. \tag{1.1}$$

Note that in case of minimal submanifolds of Euclidean spaces, this inequality gives a new condition on the curvature of the minimal submanifold. The inequality is proved in [C] using only the Gauss equation and an algebraic lemma. For minimal submanifolds, in the same paper it is proved that equality holds in (1.1) at some point if and only if the index of relative nullity at that point is at least $n-2$.

For minimal submanifolds of $S^m(1)$, this inequality gives an upper bound for δ_M and reduces to

$$\delta_M \leq \tfrac{1}{2}(n+1)(n-2).$$

We say that a minimal submanifold M^n of $S^m(1)$ satisfies Chen's equality if

$$\delta_M = \tfrac{1}{2}(n+1)(n-2).$$

We will observe that any 3-dimensional totally real submanifold M of $S^6(1)$ which is contained in some totally geodesic $S^5(1)$ satisfies Chen's equality, i.e. M satisfies $\delta_M = 2$. This is not the case if M is linearly full. Under the extra assumption that either the scalar curvature is constant or certain distributions on M are integrable, a classification of all 3-dimensional totally real submanifolds satisfying Chen's equality was obtained in [CDVV1] and [CDVV2]. Here we will give a local classification of all 3-dimensional totally real submanifolds lying linearly full in $S^6(1)$ and satisfying Chen's equality, see Theorem 2, 3 and 5.

This paper is organized as follows. In Section 2, we recall some basic facts for the almost Kähler structure on $S^6(1)$, while in Section 3 we recall from [CDVV1] and [CDVV2] necessary and sufficient conditions for a 3-dimensional totally real submanifold of $S^6(1)$ to satisfy Chen's equality. Then, in Section 4, we investigate the relation between holomorphic curves in $\mathbb{C}P^2(4)$ and totally real submanifolds of $S^6(1)$. More specifically, we prove

Theorem 1. *Let $\phi : N_1 \longrightarrow \mathbb{C}P^2(4)$ be a holomorphic curve in $\mathbb{C}P^2(4)$. Let PN_1 be the circle bundle over N_1 induced by the Hopf fibration $p : S^5(1) \to \mathbb{C}P^2(4)$ and let ψ be the isometric immersion such that the following diagram commutes:*

$$\begin{array}{ccc} PN_1 & \xrightarrow{\psi} & S^5(1) \\ \downarrow & & \downarrow p \\ N_1 & \xrightarrow{\phi} & \mathbb{C}P^2(4) \end{array}$$

Then, there exists a totally geodesic embedding i of $S^5(1)$ into the nearly Kähler 6-sphere such that the immersion $i \circ \psi : PN_1 \to S^6(1)$ is a 3-dimensional totally real immersion in $S^6(1)$ satisfying Chen's equality.

Similarly, in Section 5, we first recall the basics about almost complex curves in $S^6(1)$. Then we prove the following theorem.

Theorem 2. *Let $\bar\phi : N_2 \longrightarrow S^6(1)$ be an almost complex curve (with second fundamental form α) without totally geodesic points. Denote by UN_2 the unit tangent bundle over N_2 and define a map*

$$\bar\psi : UN_2 \to S^6(1) : v \mapsto \bar\phi_\star(v) \times \tfrac{\alpha(v,v)}{\|\alpha(v,v)\|}. \tag{1.2}$$

Then $\bar\psi$ is a (possibly branched) totally real immersion into $S^6(1)$ satisfying Chen's equality. Moreover, the immersion is linearly full in $S^6(1)$.

We will show that the immersion (1.2) can be seen as a tube of radius $\pi/2$ over $\bar\phi$, in the sense of [E2]. Indeed, we will show that, fixing the point p, the unit circle in the tangent plane will be mapped into a unit circle in a plane, perpendicular to the 4-dimensional first osculating space of the almost complex surface.

In case a nontotally geodesic almost complex curve is branched or has totally geodesic points, then, using the standard theory of harmonic sequences (see a.o. [BW]), it is still possible to define plane bundles L_0 and L_1 over N_2 such that L_0 and L_1 correspond respectively to $\bar\phi_\star(TN_2)$ and the first normal space, except at the (isolated) branch points and (also isolated) totally geodesic points. This allows us to extend the immersion $\bar\psi$ to branch points and totally geodesic points. In this way we define a tube SN_2 of radius $\pi/2$ over N_2 in the direction of $(L_0 \oplus L_1)^\perp$. A small adaptation of the proof of Theorem 2 also proves the following.

Theorem 3. *Let $\bar\phi : N_2 \longrightarrow S^6(1)$ be a (branched) almost complex immersion. Then, SN_2 is a 3-dimensional (possibly branched) totally real submanifold of $S^6(1)$ satisfying Chen's equality.*

Finally, in Section 6, we prove a local converse of the above theorems.

Theorem 4. *Let $F : M^3 \to S^6(1)$ be a totally real immersion which is not linearly full in $S^6(1)$. Then M^3 automatically satisfies Chen's equality and there exists a totally geodesic $S^5(1)$, and a holomorphic immersion $\phi : N_1 \to \mathbb{C}P^2(4)$ such that F is congruent to ψ, which is obtained from ϕ as in Theorem 1.*

Theorem 5. *Let $F : M^3 \to S^6(1)$ be a linearly full totally real immersion of a 3-dimensional manifold satisfying Chen's equality. Let p be a non totally geodesic point of M^3. Then there exists a (possibly branched) almost complex curve $\bar\phi : N_2 \to S^6(1)$ such that F is locally around p congruent to $\bar\psi$, which is obtained from $\bar\phi$ as in Theorem 3.*

2. The vector cross product and the almost complex structure on $S^6(1)$

We give a brief exposition of how the standard nearly Kähler structure on $S^6(1)$ arises in a natural manner from Cayley multiplication. For further details about the Cayley numbers and their automorphism group G_2, we refer the reader to [W] and [HL].

The multiplication on the Cayley numbers $\mathcal{O}$ may be used to define a vector cross product on the purely imaginary Cayley numbers $\mathbb{R}^7$ using the formula

$$u \times v = \tfrac{1}{2}(uv - vu), \tag{2.1}$$

while the standard inner product on $\mathbb{R}^7$ is given by

$$(u, v) = -\tfrac{1}{2}(uv + vu). \tag{2.2}$$

It is now elementary [HL] to show that

$$u \times (v \times w) + (u \times v) \times w = 2(u, w)v - (u, v)w - (w, v)u, \tag{2.3}$$

and that the triple scalar product $(u \times v, w)$ is skew symmetric in u, v, w.

Conversely, Cayley multiplication on $\mathcal{O}$ is given in terms of the vector cross product and the inner product by

$$(r+u)(s+v) = rs - (u, v) + rv + su + (u \times v), \quad r, s \in \mathrm{Re}(\mathcal{O}),\ u, v \in \mathrm{Im}(\mathcal{O}). \tag{2.4}$$

In view of (2.1), (2.2) and (2.4), it is clear that the group G_2 of automorphisms of $\mathcal{O}$ is precisely the group of isometries of $\mathbb{R}^7$ preserving the vector cross product.

An ordered orthonormal basis $e_1, \dots, e_7$ is said to be a G_2-frame if

$$e_3 = e_1 \times e_2, \quad e_5 = e_1 \times e_4, \quad e_6 = e_2 \times e_4, \quad e_7 = e_3 \times e_4. \tag{2.5}$$

For example, the standard basis of $\mathbb{R}^7$ is a G_2-frame. Moreover, if e_1, e_2, e_4 are mutually orthogonal unit vectors with e_4 orthogonal to $e_1 \times e_2$, then e_1, e_2, e_4 determine a unique G_2-frame $e_1, \dots, e_7$ and $(\mathbb{R}^7, \times)$ is generated by e_1, e_2, e_4 subject to the relations

$$e_i \times (e_j \times e_k) + (e_i \times e_j) \times e_k = 2\delta_{ik}e_j - \delta_{ij}e_k - \delta_{jk}e_i. \tag{2.6}$$

Therefore, for any G_2-frame, we have the following multiplication table [W].

$\times$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	0	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	$-e_3$	0	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	0	e_7	$-e_6$	e_5	$-e_4$
e_4	$-e_5$	$-e_6$	$-e_7$	0	e_1	e_2	e_3
e_5	e_4	$-e_7$	e_6	$-e_1$	0	$-e_3$	e_2
e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	0	$-e_1$
e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	0

The standard nearly Kähler structure on $S^6(1)$ is then obtained as follows:

$$Ju = x \times u, \qquad u \in T_xS^6(1),\ x \in S^6(1).$$

It is clear that J is an orthogonal almost complex structure on $S^6(1)$. In fact J is a nearly Kähler structure in the sense that the (2,1)-tensor field G on $S^6(1)$, defined by

$$G(X, Y) = (\widetilde{\nabla}_X J)(Y),$$

where $\widetilde{\nabla}$ is the Levi-Civita connection on $S^6(1)$, is skew-symmetric. A straightforward computation also shows that

$$G(X, Y) = X \times Y - \langle x \times X, Y \rangle\, x.$$

For more information on the properties of $\cdot$, J and G, we refer to [Ca2], [BVW] and [DVV2].

It is well known (see for instance [B, page 32] or [DV]) that the complex structure of $\mathbb{C}^3$ induces a Sasakian structure (φ, ξ, η, g) on $S^5(1)$ starting from $\mathbb{C}^3$. Its relation with the vector cross product on $\mathbb{R}^7$ is less known, and therefore we sketch it here.

We consider $S^5(1)$ as the hypersphere in $S^6(1) \subset \mathbb{R}^7$ given by the equation $x_4 = 0$ and define

$$j : S^5(1) \to \mathbb{C}^3 : (x_1, x_2, x_3, 0, x_5, x_6, x_7) \mapsto (x_1 + ix_5, x_2 + ix_6, x_3 + ix_7).$$

Then at a point $p = (x_1, \ldots, x_7)$ the structure vector field ξ is given by

$$\xi(p) = (x_5, x_6, x_7, 0, -x_1, -x_2, -x_3) = e_4 \times p.$$

And for a tangent vector $v = (v_1, v_2, v_3, 0, v_5, v_6, v_7)$, orthogonal to ξ, we have that the endomorphism φ of the Sasakian structure satisfies

$$\varphi(v) = (-v_5, -v_6, -v_7, 0, v_1, v_2, v_3) = v \times e_4.$$

Since $\varphi\xi(p) = 0 = (e_4 \times p) \times e_4 - \langle (e_4 \times p) \times e_4, p\rangle\, p$, we deduce that for any tangent vector w to $S^5(1)$

$$\varphi(w) = w \times e_4 - \langle w \times e_4, p\rangle\, p. \tag{2.7}$$

Following [YI], we call a submanifold M^n of S^5 invariant if $\varphi(T_pM) \subset T_pM$ for every p. If n is odd, then ξ is automatically tangent to M. Assume $n = 3$. The Hopf fibration $p : S^5(1) \to \mathbb{C}P^2(4)$ annihilates ξ, i.e. $dp(\xi) = 0$. Then if M^3 is invariant, $p(M^3)$ is a holomorphic curve. Conversely, let $\phi : N_1 \longrightarrow \mathbb{C}P^2(4)$ be a holomorphic curve and let PN_1 be the circle bundle over N_1 induced by the Hopf fibration and let ψ be the immersion such that the following diagram commutes:

$$\begin{array}{ccc} PN_1 & \xrightarrow{\psi} & S^5(1) \\ \big\downarrow & & \big\downarrow p \\ N_1 & \xrightarrow{\phi} & \mathbb{C}P^2(4) \end{array} \tag{2.8}$$

Then ψ is an invariant immersion in the Sasakian space form $S^5(1)$ with structure vector field ξ tangent along ψ.

3. Totally real submanifolds and Chen's equality

As we stated in the introduction, a minimal submanifold M^3 of S^n satisfies Chen's equality if and only if its index of relative nullity is at least 1 everywhere. More precisely

Lemma 3.1. [C] *Let M be a 3-dimensional minimal submanifold of $S^n(1)$ with second fundamental form h. Then $\delta_M(p) = 2$ if and only if there exist a tangent vector v of T_pM such that $h(v, w) = 0$ for all $w \in T_pM$.*

Next, we restrict ourselves to totally real immersions $F : M^3 \to S^6(1)$. We call M^3 totally real if the almost complex structure J maps the tangent space into the normal space. In [E1] Ejiri proved that a 3-dimensional totally real submanifold of $S^6(1)$ is orientable and minimal and that $G(X,Y)$ is orthogonal to M, for tangent vectors X and Y. We denote the Levi-Civita connection of M by ∇. The formulas of Gauss and Weingarten are then respectively given by

$$\widetilde{\nabla}_X F_\star Y = F_\star(\nabla_X Y) + h(X,Y), \tag{3.1}$$

$$\widetilde{\nabla}_X \eta = -F_\star(A_\eta X) + \nabla^\perp_X \eta, \tag{3.2}$$

for tangent vector fields X and Y and normal vector field η. The second fundamental form h is related to A_η by $\langle h(X,Y),\eta\rangle = \langle A_\eta X, Y\rangle$. From (3.1) and (3.2), we find that

$$\nabla^\perp_X JF_\star(Y) = JF_\star(\nabla_X Y) + G(F_\star X, F_\star Y), \tag{3.3}$$

$$F_\star(A_{JY}X) = -Jh(X,Y). \tag{3.4}$$

The above formulas immediately imply that

$$\langle h(X,Y), JF_* Z\rangle = \langle h(X,Z), JF_* Y\rangle\,, \tag{3.5}$$

i.e. $\langle h(X,Y), JF_*Z\rangle$ is totally symmetric.

Let us now assume that M^3 is a 3-dimensional totally real submanifold with $\delta_M = 2$. Then, from [CDVV1] we obtain the following characterization:

Lemma 3.2. *Let $F : M^3 \to S^6(1)$ be a totally real isometric immersion with $\delta_M = 2$. Then in a neighborhood of every nontotally geodesic point p, there exist local orthonormal vector fields E_1, E_2, E_3 and a function λ such that*

$$\begin{aligned} h(E_1,E_1) &= \lambda JF_\star E_1 \qquad & h(E_1,E_3) &= 0,\\ h(E_1,E_2) &= -\lambda JF_\star E_2 \qquad & h(E_2,E_3) &= 0,\\ h(E_2,E_2) &= -\lambda JF_\star E_1 \qquad & h(E_3,E_3) &= 0,\end{aligned}$$

where $2\lambda^2 = 3-\tau$. Moreover $G(F_\star E_1, F_\star E_2) = JF_\star E_3$, $G(F_\star E_2, F_\star E_3) = JF_\star E_1$ and $G(F_\star E_3, F_\star E_1) = JF_\star E_2$.

Now, let $p \in M$ and let $\{E_1, E_2, E_3\}$ be the local orthonormal frame as defined in Lemma 3.2, then we get from the Codazzi equations (see Lemma 6.3 of [CDVV2]) that there exist local functions a, b, c, d such that

$$\begin{aligned} &\nabla_{E_1}E_1 = cE_2 + aE_3, && \nabla_{E_1}E_2 = -cE_1 + bE_3, && \nabla_{E_1}E_3 = -aE_1 - bE_2,\\ &\nabla_{E_2}E_1 = dE_2 - bE_3, && \nabla_{E_2}E_2 = -dE_1 + aE_3, && \nabla_{E_2}E_3 = bE_1 - aE_2,\\ &\nabla_{E_3}E_1 = -\tfrac{1}{3}(b+1)E_2, && \nabla_{E_3}E_2 = \tfrac{1}{3}(b+1)E_1, && \nabla_{E_3}E_3 = 0,\end{aligned}$$

and moreover

$$E_1(\lambda) = -3\lambda d, \quad E_2(\lambda) = 3\lambda c, \quad E_3(\lambda) = \lambda a.$$

4. Proof of Theorem 1

Let $\phi : N_1 \longrightarrow \mathbb{C}P^2(4)$ be a holomorphic curve. We use the same notation as in (2.8), lift ϕ to an invariant immersion $\psi : PN_1 \to S^5$, but for simplicity we identify PN_1 with its image in $S^5(1)$. If we denote the second fundamental form of the immersion ψ by h, the basic formulas (see [YI]) for such submanifolds imply that $h(X, \xi) = 0$ for any tangent vector field X. Hence by Lemma 3.1, PN_1 is a minimal 3-dimensional submanifold of $S^5(1)$ satisfying Chen's equality.

Now, in order to prove Theorem 1, we still have to show that there exists an embedding i of $S^5(1)$ into $S^6(1)$ such that $i \circ \psi$ is a totally real immersion. In order to do so, we take the embedding of $S^5(1)$ into $S^6(1)$ as given in Section 2. We take a unit normal vector x, which is orthogonal to ξ at a point $p \in PN_1$. Then $\{x, \varphi x, \xi\}$ is a local orthonormal basis for the tangent space. Applying the formulas of Section 2, we notice that $\xi = e_4 \times p$ and hence $J\xi = p \times (e_4 \times p) = e_4$ is orthogonal to $S^5(1)$ and thus also orthogonal to PN_1. Since $\varphi x = x \times e_4 - \langle x \times e_4, p \rangle\, p$, we get that

$$\begin{aligned}
&\langle p \times x, x \rangle = 0,\\
&\langle p \times x, \xi \rangle = \langle p \times x, e_4 \times p \rangle = -\langle x, e_4 \rangle = 0,\\
&\langle p \times x, \varphi x \rangle = \langle p \times x, x \times e_4 \rangle = -\langle p, e_4 \rangle = 0.
\end{aligned}$$

Hence Jx is also orthogonal to $\psi_* T_p PN_1$. Since this holds for any vector x orthogonal to ξ and $J\xi$ itself is also orthogonal to $\psi_* T_p PN_1$, we get that PN_1 is totally real.

5. Almost complex surfaces and Chen's equality

An immersion $\bar\phi : N_2 \to S^6(1)$ is called almost complex if J preserves the tangent space, i.e. $J_p \bar\phi_\star(T_p N_2) = \bar\phi_\star(T_p N_2)$. It is well known that such immersions are always minimal and, as indicated in [BVW], there are essentially 4 types of almost complex immersions in $S^6(1)$; namely, those which are

(I) linearly full in $S^6(1)$ and superminimal,
(II) linearly full in $S^6(1)$ but not superminimal,
(III) linearly full in some totally geodesic $S^5(1)$ in $S^6(1)$ (and thus by [Ca1] necessarily not superminimal),
(IV) totally geodesic.

Setting aside the trivial case (IV) (which consists of surfaces of the form $S^6(1) \cap V$ where $V \subset \operatorname{Im} \mathcal{O}$ is an associative 3-plane [Br]), case (I) has been previously studied by Bryant [Br] who gave a "Weierstrass representation" theorem for such immersions and proved that there are compact almost complex curves of type (I) of any genus. The case of genus 0 has also been considered by Ejiri [E2] who described all S^1-symmetric examples, and by Sekigawa [S] who showed that if an almost complex curve of genus zero has constant curvature K, then $K = 1$ or $\frac{1}{6}$ and gave examples of each case.

Almost complex curves of type (III) have been studied in [BVW] where they are related to totally real minimal immersions of surfaces in $\mathbb{C}P^2(4)$; almost complex curves of type (II) were completely described in the special case when N_2 is a torus in [BPW].

Now, let $\bar\phi : N_2 \longrightarrow S^6(1)$ be an almost complex curve. We denote its position vector in $\mathbb{R}^7$ also by $\bar\phi$. For proofs of elementary properties of such surfaces, we refer to [S]. Here, we simply recall that for tangent vector fields X and Y to M and for a normal vector field η, we have

$$\alpha(X, JY) = J\alpha(X, Y), \tag{5.1}$$

$$A_{J\eta} = JA_\eta = -A_\eta J, \tag{5.2}$$

$$\nabla^\perp_X J\eta = G(X, \eta) + J\nabla^\perp_X \eta, \tag{5.3}$$

$$(\nabla\alpha)(X, Y, JZ) = J(\nabla\alpha)(X, Y, Z) + G(\bar{\phi}_\star X, \alpha(Y, Z)), \tag{5.4}$$

where α denotes the second fundamental form of the immersion and the pull-back of J to N_2 is also denoted by J.

Next, if necessary, by restricting ourselves to an open dense subset of N_2, we may assume that N_2 does not contain any totally geodesic points. Let $p \in N_2$ and let V be an arbitrary unit tangent vector field defined on a neighborhood W of p. We define a nonzero function $\mu = \|\alpha(V, V)\|$, which does not depend on the choice of V. Let $U = JV$, then, using the properties of the vector cross product, it is easy to see that $F_1 = \bar{\phi}$, $F_2 = \bar{\phi}_\star V$, $F_3 = J\bar{\phi}_\star V$, $F_4 = \alpha(V, V)/\mu$, $F_5 = \alpha(V, JV)/\mu = J\alpha(V, V)/\mu = F_1 \times F_4$, $F_6 = F_2 \times \alpha(V, V)/\mu$ and $F_7 = F_3 \times \alpha(V, V)/\mu$ form a G_2-frame and hence satisfy the multiplication table as defined in Section 2.

Since $F_4, \ldots, F_7$ form a basis for the normal space along N_2, it is clear that we can write any normal vector field as a linear combination of these basis vector fields. Thus there exist functions $a_1, \ldots, a_4$ such that

$$(\nabla\alpha)(V, V, V) = \mu(a_1F_4 + a_2F_5 + a_3F_6 + a_4F_7). \tag{5.5}$$

Then using (5.4) and the multiplication table, we get that

$$\begin{aligned}(\nabla\alpha)(V, V, U) &= \mu(a_1F_1 \times F_4 + a_2F_1 \times F_5 + a_3F_1 \times F_6 + a_4F_1 \times F_7) + \mu F_6 \\ &= \mu(-a_2F_4 + a_1F_5 + (1 + a_4)F_6 - a_3F_7).\end{aligned} \tag{5.6}$$

From (5.5) and (5.6) it is immediately clear that

1. N_2 is an almost complex surface of Type (I) if and only if $a_3 = 0$ and $a_4 = -\frac{1}{2}$.
2. N_2 is an almost complex surface of Type (III) if and only if $a_4 + a_3^2 + a_4^2 = 0$.

Indeed, N_2 is superminimal if and only if $(\nabla\alpha)(V, V, V)$ and $(\nabla\alpha)(V, V, U)$ are orthogonal, which proves the first claim. The second claim can be proved by noting that N_2 is not linearly full if and only if the components of $(\nabla\alpha)(V, V, V)$ and $(\nabla\alpha)(V, V, U)$ orthogonal to F_4 and F_5 are proportional (the "if" part can be proved using the theory of harmonic sequences, the "only if" part is obvious).

Now, we introduce local functions μ_1 and μ_2 on N_2 by

$$\begin{aligned}\nabla_V V &= \mu_1 U, \qquad & \nabla_U U &= \mu_2 V,\\ \nabla_V U &= -\mu_1 V, \qquad & \nabla_U V &= -\mu_2 U.\end{aligned}$$

Also, we can still express a_1 and a_2 in terms of the function μ. From (5.5) it follows that

$$\begin{aligned}a_1\mu^2 &= \langle(\nabla\alpha)(V, V, V), \alpha(V, V)\rangle \\ &= \tfrac{1}{2}V\,\langle\alpha(V, V), \alpha(V, V)\rangle - 2\mu_1\,\langle\alpha(V, U), \alpha(V, V)\rangle \\ &= \tfrac{1}{2}V(\mu^2) = \mu V(\mu).\end{aligned}$$

Hence $a_1 = \frac{V(\mu)}{\mu}$. Similarly, we obtain $a_2 = -\frac{U(\mu)}{\mu}$.

Now, in order to construct explicitly the totally real immersion from the unit tangent bundle, we still need a technical lemma.

Lemma 5.1. *Denote by D the standard connection on $\mathbb{R}^7$. Then, we have*

$$\begin{aligned}
D_V(\mu F_4) &= \mu(-\mu F_2 + a_1F_4 + (a_2+2\mu_1)F_5 + a_3F_6 + a_4F_7),\\
D_U(\mu F_4) &= \mu(\mu F_3 - a_2F_4 + (a_1-2\mu_2)F_5 + (1+a_4)F_6 - a_3F_7),\\
D_V(\mu F_5) &= \mu(-\mu F_3 - (a_2+2\mu_1)F_4 + a_1F_5 + (1+a_4)F_6 - a_3F_7),\\
D_U(\mu F_5) &= \mu(-\mu F_2 - (a_1-2\mu_2)F_4 - a_2F_5 - a_3F_6 - a_4F_7),\\
D_V(\mu F_6) &= \mu(-a_3F_4 - (a_4+1)F_5 + a_1F_6 + (a_2+3\mu_1)F_7),\\
D_U(\mu F_6) &= \mu(-(a_4+1)F_4 + a_3F_5 - a_2F_6 + (a_1-3\mu_2)F_7),\\
D_V(\mu F_7) &= \mu(-a_4F_4 + a_3F_5 - (a_2+3\mu_1)F_6 + a_1F_7),\\
D_U(\mu F_7) &= \mu(a_3F_4 + a_4F_5 + (3\mu_2-a_1)F_6 - a_2F_7).
\end{aligned}$$

Proof. We have

$$\begin{aligned}
D_V(\alpha(V,V)) &= -\bar\phi_\star(A_{\alpha(V,V)}V) + \nabla^\perp_V\alpha(V,V) - \langle V,\alpha(V,V)\rangle\,\bar\phi\\
&= -\mu^2\bar\phi_\star V + (\nabla\alpha)(V,V,V) + 2\mu_1\alpha(U,V)\\
&= -\mu^2F_2 + \mu(a_1F_4 + a_2F_5 + a_3F_6 + a_4F_7) + 2\mu_1\mu F_5\\
&= \mu(-\mu F_2 + a_1F_4 + (a_2+2\mu_1)F_5 + a_3F_6 + a_4F_7).
\end{aligned}$$

The next three equations are obtained in a similar way. Using the first 4 equations, we now get that

$$\begin{aligned}
D_V(F_2\times\alpha(V,V)) &= D_VF_2\times\alpha(V,V) + F_2\times D_V\alpha(V,V)\\
&= \mu_1U\times\alpha(V,V) + \alpha(V,V)\times\alpha(V,V) - \langle V,V\rangle\,\bar\phi\times\alpha(V,V)\\
&\qquad + F_2\times\mu(-\mu F_2 + a_1F_4 + (a_2+2\mu_1)F_5 + a_3F_6 + a_4F_7)\\
&= \mu_1\mu F_7 - \mu F_5 + \mu a_1F_6 + \mu(a_2+2\mu_1)F_7 - \mu a_3F_4 - \mu a_4F_5\\
&= \mu(-a_3F_4 - (a_4+1)F_5 + a_1F_6 + (a_2+3\mu_1)F_7).
\end{aligned}$$

The other equations are again obtained in a similar fashion. □

Proof of Theorem 2. We define a map

$$\bar\psi : U(N_2)\to S^6(1) : v\mapsto \bar\phi_\star(v)\times\tfrac{\alpha(v,v)}{\|\alpha(v,v)\|}.$$

Using the above vector fields, we can write $v=\cos(t/3)V+\sin(t/3)U$; and an easy computation shows that the map $\bar\psi$ can be locally parameterized by

$$\bar\psi(q,t) = \cos t F_6(q) + \sin t F_7(q), \tag{5.7}$$

where $q\in W$ and $t\in\mathbb{R}$. We immediately see that

$$\bar\psi_\star(\tfrac{\partial}{\partial t}) = -\sin tF_6 + \cos tF_7. \tag{5.8}$$

Using Lemma 5.1, we then obtain that

$$\begin{aligned}
\bar\psi_\star(V) &= \cos tD_VF_6 + \sin tD_VF_7\\
&= -\cos t\tfrac{V(\mu)}{\mu}F_6 + \cos t(-a_3F_4 - (a_4+1)F_5 + a_1F_6 + (a_2+3\mu_1)F_7)\\
&\quad - \sin t\tfrac{V(\mu)}{\mu}F_7 + \sin t(-a_4F_4 + a_3F_5 - (a_2+3\mu_1)F_6 + a_1F_7)\\
&= (-a_3\cos t - a_4\sin t)F_4 + (a_3\sin t - (a_4+1)\cos t)F_5\\
&\qquad + (3\mu_1 - \tfrac{U(\mu)}{\mu})\bar\psi_\star(\tfrac{\partial}{\partial t}).
\end{aligned}\tag{5.9}$$

Using similar computations, we also get that

$$\begin{aligned}\bar\psi_\star(U) = (a_3 \sin t - &(1+a_4)\cos t)F_4 + (a_3 \cos t + a_4 \sin t)F_5\\ &+(-3\mu_2 + \tfrac{V(\mu)}{\mu})\bar\psi_\star(\tfrac{\partial}{\partial t}).\end{aligned} \tag{5.10}$$

From (5.8), (5.9) and (5.10), we see that $\bar\psi$ is an immersion at points (q, t) satisfying

$$(a_3(q)\cos t + a_4(q)\sin t)^2 + (a_3(q)\sin t - (1 + a_4(q))\cos t)^2 \neq 0, \tag{5.11}$$

which is clearly a dense open subset of UN_2. Actually, if N_2 is an almost complex surface of Type I, we get that (5.11) is always satisfied. More precisely, by choosing the vector field V such that at some arbitrary point q the function $\|(\nabla\alpha)(u, u, u)\|^2$, defined on the unit circle in the tangent plane takes an absolute maximal value at $V(q)$, we can make sure that $a_3(q) = 0$ and $a_4(q) \neq 0$. Then the condition (5.11) holds unless $a_4 = -1$ and $\sin(t) = 0$. Clearly this means that the branching points of the immersion ψ_* are two antipodal points on the circle corresponding to points on N_2 where the second normal space does not have maximal rank.

Let us now restrict to the open dense subset on which $\bar\psi$ is an immersion. Then, we get from (5.8)–(5.10) that $-\sin tF_6 + \cos tF_7$, F_4 and F_5 form a basis of $\bar\psi_\star(UN_2)$ along UN_2. Then, using the multiplication table in Section 2 and the definition of the almost complex structure on $S^6(1)$, we see that for the mapping $\bar\psi$:

$$\begin{aligned}&J(-\sin tF_6 + \cos tF_7) = (\cos tF_6 + \sin tF_7) \times (-\sin tF_6 + \cos tF_7) = -F_1\\ &J(F_4) = (\cos tF_6 + \sin tF_7) \times F_4 = -\cos tF_2 - \sin tF_3,\\ &J(F_5) = (\cos tF_6 + \sin tF_7) \times F_5 = -\sin tF_2 + \cos tF_3,\end{aligned}$$

from which we deduce that $\bar\psi$ is a totally real (and hence minimal) immersion in $S^6(1)$. To show that $\bar\psi$ satisfies Chen's equality, we notice that

$$D_{\frac{\partial}{\partial t}}\bar\psi_\star(\tfrac{\partial}{\partial t}) = -\cos tF_6 - \sin tF_7 = -\psi, \tag{5.12}$$

$$D_{\frac{\partial}{\partial t}}F_4 = 0, \tag{5.13}$$

$$D_{\frac{\partial}{\partial t}}F_5 = 0. \tag{5.14}$$

Hence, using Lemma 3.1, we see that UN_2 (with its induced metric) satisfies Chen's equality.

In order to show that UN_2 lies linearly full in $S^6(1)$, we compute the first normal space of the immersion $\bar\psi$. Using (5.12)–(5.14), it is clear this normal space is obtained by taking the normal components (tangential to $S^6(1)$) of D_VF_4, D_UF_4, $D_V(F_5)$ and $D_U(F_5)$. So from Lemma 5.1 we get that the first normal space is spanned by $X = -\mu F_2$ and $Y = \mu F_3$. Since $D_VF_2 = \mu_1F_3 + \mu F_4 - F_1$, we get that F_1 belongs to the second normal space and therefore $\bar\psi(UN_2)$ is linearly full in $S^6(1)$.

Remark 5.1. Let X and Y be arbitrary orthonormal vector fields (along $\bar\phi$) which are orthogonal to the tangent space and the first normal space of N_2. Then taking a different parameter t, if necessary, it follows from (5.7) that $\bar\psi$ can also be locally parameterized as

$$\bar\psi(t, q) = \cos tX(q) + \sin tY(q),$$

where $t \in \mathbb{R}$ and $q \in W$.

Remark 5.2. A straightforward but lengthy computation shows that the immersion $\bar{\psi}$ has constant scalar curvature if and only if the immersion $\bar{\phi}$ has constant Gaussian curvature and is superminimal. It then follows that Example 3.1 of [CDVV1] is obtained, as in Theorem 2, from the almost complex surface of constant curvature $\frac{1}{6}$ in $S^6(1)$.

Remark 5.3. Using the above formulas, one can compute that on UN_2, the relative nullity $\mathcal{D}$ satisfies $\mathcal{D} = \mathrm{span}\{\frac{\partial}{\partial t}\}$ and that $\mathcal{D}^\perp$ is integrable if and only if N_2 is an almost complex surface of Type III (i.e. $\bar{\phi}(N_2)$ is linearly full in a totally geodesic $S^5(1)$). Hence the examples of [CDVV2] correspond to almost complex surfaces of Type III.

Remark 5.4. In the spirit of [E2], a tube with radius r around a surface immersion $\phi : M \to S^6(1)$ (or any other surrounding space) in the direction of a plane bundle B along ϕ is defined as the image of the mapping

$$UB \to S^6(1) : v \mapsto \exp_{\phi(p)} rv = \cos(r)\phi(p) + \sin(r)v,$$

where $UB = \{v \in B \mid \|v\| = 1\}$. From (5.7) we immediately obtain that $\bar{\psi}$ is a tube with radius $\pi/2$ the direction of the plane bundle determined by F_6 and F_7, i.e. the orthogonal complement of the first osculating space. We remark that the case when the radius is $\frac{\pi}{2}$, which we need here, is not considered in [E2].

Remark 5.5. If $\bar{\phi}$ is branched or has totally geodesic points we can still define the tube as mentioned in the introduction, but we have no control on the points where the tube is not immersed. We only know they lie on the circle corresponding to branch points, totally geodesic points or points where the second normal space does not have maximal rank.

6. The proofs of Theorem 4 and Theorem 5

Proof of Theorem 4. First, we consider the case that $\psi : M^3 \to S^6(1)$ is a totally real immersion of a 3-dimensional submanifold M^3 in $S^6(1)$ which is not linearly full in $S^6(1)$. We identify M with its image in $S^6(1)$. Then, there exists a constant unit vector u in $\mathbb{R}^7$ which is orthogonal to the image of M. By applying an element of G_2, we may assume that $u = e_4$ so that we can apply the formulas of Section 2. Since M is totally real and e_4 is normal to M, $\xi = e_4 \times p$ is a tangential vector field. Therefore, we can again consider the following diagram:

$$\begin{array}{ccc} M & \xrightarrow{\psi} & S^5(1) \subset S^6(1) \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ N_1 & \xrightarrow{\phi} & \mathbb{C}P^2(4) \end{array},$$

where p denotes the Hopf fibration and $N_1 = p(M)$ is well-defined since ξ is tangential to M.

To prove that ϕ is a holomorphic immersion in $\mathbb{C}P^2(4)$, it is sufficient to show that M is an invariant submanifold of the Sasakian space form $S^5(1)$. In order to do so, let X be a unit tangent vector field to M which is orthogonal to $\xi = -Je_4$. Then, since $G(U,V)$ is normal to M for tangent vector fields U and V on M, we get that the normal space is spanned by e_4, JX and $G(X,\xi) = Je_4 \times X$. Then, from (2.7), we recall that

$$\varphi(X) = X \times e_4 - \langle X \times e_4, p\rangle\, p = X \times e_4.$$

Since

$$\begin{aligned}
&\langle \varphi(X), JX\rangle = \langle X \times e_4, p \times X\rangle = -\langle p, e_4\rangle = 0,\\
&\langle \varphi(X), e_4\rangle = \langle X \times e_4, e_4\rangle = 0,\\
&\langle \varphi(X), (p \times e_4) \times X\rangle = \langle X \times e_4, (p \times e_4) \times X\rangle = 0,
\end{aligned}$$

we obtain that M^3 is an invariant submanifold of $S^5(1)$. □

Remark 6.1. For totally real submanifolds as in the previous theorem, it is well-known that M^3 has constant scalar curvature if and only if the corresponding holomorphic curve in $\mathbb{C}P^2(4)$ has constant Gaussian curvature, i.e. if and only if N_2 is totally geodesic (in which case the lift is again totally geodesic) or if N_2 has constant curvature 2 (in which case a straightforward computation shows that M is locally isometric with Example 3.2 of [CDVV1]).

Proof of Theorem 5. Let p be a nontotally geodesic point of M^3. We take the local frame recalled in Section 3. Using [Sp, Vol 1, p. 204], we can identify a neighborhood of p with a neighborhood $I \times W_1$ of the origin in $\mathbb{R}^3$ (with coordinates (t,u,v)) such that $p = (0,0,0)$ and the vector field $E_3 = \frac{\partial}{\partial t}$. Then there exist functions α_1 and α_2 on W_1 such that $E_1 + \alpha_1 E_3$ and $E_2 + \alpha_2 E_3$ form a basis for the tangent space to $W_1 \subset M^3$ at the point $q = (0,u,v)$.

First, we consider the case when $a = 0$ and $b = -1$ on a neighborhood of the point p. In this case, using Lemma 2.4, we obtain

$$\begin{aligned}
&\tilde\nabla_{E_1} JF_\star E_3 = J\tilde\nabla_{E_1} F_\star E_3 - JF_\star E_2 = -aJF_\star E_1 - (b+1)JF_\star E_2 = 0,\\
&\tilde\nabla_{E_2} JF_\star E_3 = J\tilde\nabla_{E_2} F_\star E_3 + JF_\star E_1 = (b+1)JF_\star E_1 - aJF_\star E_2 = 0,\\
&\tilde\nabla_{E_3} JF_\star E_3 = J\tilde\nabla_{E_3} F_\star E_3 = 0.
\end{aligned}$$

Hence, $JF_\star E_3$ is a constant vector along M^3. Since the first normal space is spanned by $JF_\star E_1$ and $JF_\star E_2$ and since

$$\begin{aligned}
&\left\langle \tilde\nabla_{E_i} JF_\star E_1, JF_\star E_3\right\rangle = -\left\langle JF_\star E_1, \tilde\nabla_{E_i} JF_\star E_3\right\rangle = 0\\
&\left\langle \tilde\nabla_{E_i} JF_\star E_2, JF_\star E_3\right\rangle = -\left\langle JF_\star E_2, \tilde\nabla_{E_i} JF_\star E_3\right\rangle = 0,
\end{aligned}$$

Erbacher's theorem ([Er]) implies that M^3 is contained in totally geodesic $S^5(1)$ in $S^6(1)$, which contradicts our assumption.

Next we assume that $a = 0$ and $b = -1$ on a open set W_2 of W_1. But then still $JF_\star E_3$ is a constant vector along W_2 and since $D_{\frac{\partial}{\partial t}} JF_\star E_3 = 0$, we obtain that $JF_\star E_3$ is constant on an open set of M^3 containing W_2. This leads to a contradiction as above.

Therefore the set W of nontotally geodesic points such that $a^2 + (b+1)^2 \neq 0$ is an open dense subset of M and $W \cap W_1$ is open and dense in W_1.

Now since $\nabla_{E_3} E_3 = 0$ and $h(E_3, E_3) = 0$, it is clear that $F(M)$ can be reconstructed from W_1 by

$$F(t,u,v) = \cos t F(0,u,v) + \sin t F_\star(E_3(0,u,v)) \tag{6.1}$$

Now, we define another mapping by

$$\bar{\phi} : W_1 \to S^6(1) : (u,v) \mapsto JF_\star(E_3(0,u,v)).$$

Then, we have for $q = (0,u,v) \in W_1$:

$$\bar{\phi}(q) = JF_\star E_3(q) = F(q) \times F_\star E_3(q), \tag{6.2}$$

$$\begin{aligned} \bar{\phi}_\star(E_1 + \alpha_1 E_3) &= D_{E_1 + \alpha_1 E_3} JF_\star E_3 = -a(q) JF_\star E_1(q) - (b+1)(q) JF_\star E_2(q) \\ &= -a(q) F(q) \times F_\star E_1(q) - (b+1)(q) F(q) \times F_\star E_2(q), \end{aligned} \tag{6.3}$$

$$\begin{aligned} \bar{\phi}_\star(E_2 + \alpha_2 E_3) &= D_{E_2 + \alpha_2 E_3} JF_\star E_3 = ((b+1) JF_\star E_1 - a JF_\star E_2)(q) \\ &= (b+1)(q) F(q) \times F_\star E_1(q) - a(q) F(q) \times F_\star E_2(q). \end{aligned} \tag{6.4}$$

Hence $\bar{\phi}$ is an immersion at points where $a(q) \neq 0$ or $b(q) \neq -1$ and then $\bar{\phi}_\star(T_q W_1)$ is spanned by $JF_\star E_1(q)$ and $JF_\star E_2(q)$. Since

$$\begin{aligned} (F \times F_\star E_3) \times (F \times F_\star E_1) &= (F \times (F \times F_\star E_3)) \times F_\star E_1 \\ &= F_\star E_1 \times F_\star E_3 = -F \times F_\star E_2 \\ (F \times F_\star E_3) \times (F \times F_\star E_2) &= (F \times (F \times F_\star E_3)) \times F_\star E_2 \\ &= F_\star E_2 \times F_\star E_3 = F \times F_\star E_1, \end{aligned}$$

the equations (6.2), (6.3) and (6.4) now imply that

$$\begin{aligned} \bar{\phi} \times \bar{\phi}_\star(E_1 + \alpha_1 E_3) &= (F \times F_\star E_3) \times (-aF \times F_\star E_1 - (b+1) F \times F_\star E_2) \\ &= aF \times F_\star E_2 - (b+1) F \times F_\star E_1 \\ &= -\bar{\phi}_\star(E_2 + \alpha_2 E_3). \end{aligned}$$

Hence, $\bar{\phi}$ is a (possibly branched) almost complex immersion in $S^6(1)$.

Therefore, in order to compute the first normal space, it is sufficient to compute the components normal to $\bar{\phi}(W_1)$ and tangent to $S^6(1)$ of $D_{E_1+\alpha_1 E_3} JF_\star E_1$ and $D_{E_2+\alpha_2 E_3} JF_\star E_1$. We get that

$$\begin{aligned} D_{E_1+\alpha_1 E_3} JF_\star E_1 &= (c + \alpha_1 - \tfrac{1}{3}(b+1)\alpha_1) JF_\star E_2 - \lambda F_\star E_1 + a JF_\star E_3, \\ D_{E_2+\alpha_2 E_3} JF_\star E_1 &= (d + \alpha_2 - \tfrac{1}{3}(b+1)\alpha_2) JF_\star E_2 + \lambda F_\star E_2 - (b+1) JF_\star E_3. \end{aligned}$$

Hence the first normal space to $\bar{\phi}(W_1)$ at a point q is spanned by $F_\star E_1(0,u,v)$ and $F_\star E_2(0,u,v)$. Notice that the vector fields $F(0,u,v)$ and $F_\star E_3(0,u,v)$ are mutually orthogonal vector fields which are orthogonal both to the tangent space and the first normal space of the immersion $\bar{\phi}$. Hence by Remark 5.1, and since the complex line bundles determined by $\bar{\phi}_\star(TN_2)$ and the first normal space can be extended in the points where $\bar{\phi}$ is not an immersion, the totally real immersion $\bar{\psi}$ (corresponding to $\bar{\phi}$ by Theorem 2 or 3), can be written as

$$\bar{\psi}(t,u,v) = \cos t F(0,u,v) + \sin t F_\star E_3(0,u,v),$$

such that $\bar{\psi} = F$. $\square$

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